8. Zadana je ploha $\vec{r}(u,v) = u\vec{a} + \sin u\vec{b} + v\vec{c}, u,v \in \mathbb{R}$ gdje su $\vec{a}$, $\vec{b}$ i $\vec{c}$ dati vektori.
(a) Ispitati šta su koordinatne krive.
(b) Odrediti koeficijente E, F i G prve kvadratne forme.
(c) Kada će se koordinatne krive ove plohe sijeći ortogonalno?
(d) Naći element površine $\mathrm{d}S$ dane plohe.

9. Naći površinu četverougla na helikoidu $x = a\,u\cos v$, $y = a\,u\sin v$, $z = b\,v$ (gdje su $u,v \in \mathbb{R}$), ograničenog krivima $u = 0$, $u = \frac{b}{a}$, $v = 0$, $v = 1$.

10. Naći površinu torusa $x = (a + b\cos u)\cos v$, $y = (a + b\cos u)\sin v$, $z = b\sin u$, $u,v \in [0, 2\pi]$.

11. Površ Γ definisana je vektorskom jednačinom

$$\vec{r} = (u\sin v,\, u\cos v,\, v).$$

(a) Naći prvu kvadratnu formu površi.
(b) Na površi je zadan krivoliniski trougao

$$0 \le u \le \mathrm{sh}v,\ 0 \le v \le v_0.$$

Izračunati površinu i dužine strana trougla.

12. Odrediti izraz za površinu zatvorenog područja (K) na površi $z = z(x,y)$.

14 Druga diferencijalna forma površi

Druga osnovna forma površi je

$$F_2 = (\mathrm{d}^2\vec{r}\cdot\vec{n}_0) = -(\mathrm{d}\vec{r}\cdot\mathrm{d}\vec{n}_0) = L\mathrm{d}u^2 + 2M\mathrm{d}u\mathrm{d}v + N\mathrm{d}v^2$$

gdje je

$$\mathrm{d}^2\vec{r} = \vec{r}''_{uu}\mathrm{d}u^2 + 2\vec{r}''_{uv}\mathrm{d}u\mathrm{d}v + \vec{r}''_{vv}\mathrm{d}v^2.$$

Primjetimo da vrijedi $L = \vec{n}_0\cdot\vec{r}''_{uu} = \frac{1}{W}\left(\frac{\partial\vec{r}}{\partial u}, \frac{\partial\vec{r}}{\partial v}, \frac{\partial^2\vec{r}}{\partial u^2}\right)$, $M = \vec{n}_0\cdot\vec{r}''_{uv} = \frac{1}{W}\left(\frac{\partial\vec{r}}{\partial u}, \frac{\partial\vec{r}}{\partial v}, \frac{\partial^2\vec{r}}{\partial u\partial v}\right)$, $N = \vec{n}_0\cdot\vec{r}''_{vv} = \frac{1}{W}\left(\frac{\partial\vec{r}}{\partial u}, \frac{\partial\vec{r}}{\partial v}, \frac{\partial^2\vec{r}}{\partial v^2}\right)$, odnosno koordinatno

$$L = \frac{1}{W}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \end{vmatrix}, \quad M = \frac{1}{W}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial^2 x}{\partial u\partial v} & \frac{\partial^2 y}{\partial u\partial v} & \frac{\partial^2 z}{\partial u\partial v} \end{vmatrix}, \quad N = \frac{1}{W}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \end{vmatrix}$$

Funkcije L, M i N nazivamo Gaussovim osnovnim (fundamentalnim) veličinama drugog reda ($W = \sqrt{EG - F^2}$).

13. Naći drugu kvadratnu formu za zavojnu plohu (helikoid)

$$x = au\cos v, \quad y = au\sin v, \quad z = bv, \quad u,v \in \mathbb{R}.$$

14. Pokazati: (a) da je druga kvadratna forma ravni identički jednaka nuli. (b) da je druga kvadratna forma sfere proporcionalna prvoj.

15. Naći drugu diferencijalnu formu površi zadane jednačinom $z = f(x,y)$.

16. Naći drugu kvadratnu formu za rotacionu površ $\vec{r} = (\phi(u)\cos v, \phi(u)\sin v, \psi(u))$.

\# Naći drugu kvadratnu formu za zavojnu plohu (helikoid):

$$x = au\cos v,\ y = au\sin v,\ z = bv,\ u, v \in \mathbf{R}.$$

Vektorska jednadžba plohe glasi:

$$\vec{r} = au\cos v\,\vec{i} + au\sin v\,\vec{j} + bv\,\vec{k}.$$

Tada su Gaussove veličine prvoga reda:

$$E = \left(\frac{\partial \vec{r}}{\partial u}\right)^2 = (a\cos v)^2 + (a\sin v)^2 + (0)^2 = a^2,$$

$$G = \left(\frac{\partial \vec{r}}{\partial v}\right)^2 = (au\sin v)^2 + (au\cos v)^2 + (\mathrm{b})^2 = a^2u^2 + b^2,$$

$$F = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = -a^2u\sin v\cos v + a^2u\sin v\cos u + 0 \cdot b = 0,$$

a diskriminanta prve kvadratne forme

$$W^2 = EG - F^2 = (a^2u^2 + b^2)\,a^2.$$

Kako su Gaussove veličine drugoga reda dane s:

$$L = \frac{1}{W}(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{uu}), \quad M = \frac{1}{W}(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{uv}), \quad N = \frac{1}{W}(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{vv}),$$

to računajmo mješovite produkte:

$$(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{uu}) = \begin{vmatrix} a\cos v & a\sin v & 0 \\ -au\sin v & au\cos v & b \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

$$(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{uv}) = \begin{vmatrix} a\cos v & a\sin v & 0 \\ -au\sin v & au\cos v & b \\ -a\sin v & a\cos v & 0 \end{vmatrix} = -a^2b,$$

$$(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{vv}) = \begin{vmatrix} a\cos v & a\sin v & 0 \\ -au\sin v & au\cos v & b \\ -au\cos v & -au\sin v & 0 \end{vmatrix} = 0.$$

Dakle je $L = 0, \quad M = -\dfrac{ab}{\sqrt{a^2u^2 + b^2}}, \quad N = 0,$

pa druga diferencijalna forma helikoida glasi:

$$\mathrm{II} \equiv -\frac{2ab}{\sqrt{a^2u^2 + b^2}}\,\mathrm{d}u\,\mathrm{d}v.$$

Pokazati:

1° da je druga kvadratna forma ravnine identički jednaka nuli.
2° da je druga kvadratna forma sfere proporcionalna prvoj.

1° Prema zad. 216. parametarske jednadžbe ravnine glase:

$$
\begin{aligned}
x &= x_0 + l_1 u + l_2 v \\
y &= y_0 + m_1 u + m_2 v \\
z &= z_0 + n_1 u + n_2 v \quad u, v \in \mathbf{R}.
\end{aligned}
$$

Kako je:

$$\ddot{\vec{r}}_{uu} = 0, \quad \ddot{\vec{r}}_{uv} = 0, \quad \ddot{\vec{r}}_{vv} = 0$$

to je i

$$L = \frac{1}{W}(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{uu}) = 0, \; M = \frac{1}{W}(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{uv}) = 0,$$
$$N = \frac{1}{W}(\dot{\vec{r}}_u, \dot{\vec{r}}_v, \ddot{\vec{r}}_{vv}) = 0,$$

pa je druga kvadratna forma identički jednaka nuli, tj.:

$$\mathrm{II} \equiv 0.$$

2° Prema zad. 204. i 266. vektorska jednadžba sfere glasi:

$$\vec{r} = \{ r\cos\phi\sin\theta, \; r\sin\phi\sin\theta, \; r\cos\theta \}, \; \phi \in [0, 2\pi], \; \theta \in [0, \pi],$$

a Gaussove veličine prvog reda:

$$E = \left(\frac{\partial \vec{r}}{\partial \phi}\right)^2 = r^2 \sin^2\theta, \quad G = \left(\frac{\partial \vec{r}}{\partial \theta}\right)^2 = r^2, \quad F = \frac{\partial \vec{r}}{\partial \phi} \cdot \frac{\partial \vec{r}}{\partial \theta} = 0,$$

pa je prva diferencijalna forma dana s:

$$I = r^2 \sin^2\theta \, \mathrm{d}\phi^2 + r^2 \, \mathrm{d}\theta^2.$$

Jer je $W^2 = EG - F^2 = r^4 \sin^2\theta$, to su koeficijenti druge kvadratne forme redom:

$$L = \frac{1}{W}(\dot{\vec{r}}_\phi, \dot{\vec{r}}_\theta, \ddot{\vec{r}}_{\phi\phi}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\cos\phi\sin\theta & -r\sin\phi\sin\theta & 0 \end{vmatrix} =$$

$$= \frac{r\sin\theta}{r^2\sin\theta} \begin{vmatrix} -r\sin\phi\sin\theta & r\cos\phi\sin\theta \\ -r\cos\phi\sin\theta & -r\sin\phi\sin\theta \end{vmatrix} = r\sin^2\theta.$$

$$M = \frac{1}{W}(\dot{\vec{r}}_\phi, \dot{\vec{r}}_\theta, \ddot{\vec{r}}_{\phi\theta}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\sin\phi\cos\theta & -r\cos\phi\cos\theta & 0 \end{vmatrix} = 0.$$

$$N = \frac{1}{W}(\dot{\vec{r}}_\phi, \dot{\vec{r}}_\theta, \ddot{\vec{r}}_{\theta\theta}) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\cos\phi\sin\theta & -r\sin\phi\sin\theta & -r\cos\theta \end{vmatrix} =$$

$$= \frac{r^3}{r^2\sin\theta}(\sin^2\phi\sin\theta\cos^2\theta + \cos^2\phi\sin^3\theta + \sin^2\phi\sin^3\theta +$$

$$+ \cos^2\phi\sin\theta\cos^2\theta) = r.$$

Druga diferencijalna forma dakle glasi:

$$\mathrm{II} = r\sin^2\theta\,\mathrm{d}\phi^2 + r\,\mathrm{d}\theta^2,$$

pa je zaista:

$$\mathrm{I} = r\,\mathrm{II}.$$

#
Naći drugu diferencijalnu formu plohe zadane jednadžbom:

$$z = f(x, y).$$

Prema zad. 214. i 270. imamo:

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + f(x, y)\,\vec{k},$$

$$E = 1 + p^2, \quad G = 1 + q^2, \quad F = p \cdot q,$$

$$W^2 = EG - F^2 = (1 + p^2)\;(1 + q^2) - p^2 q^2 = 1 + p^2 + q^2.$$

Gaussove veličine drugog reda jesu:

$$L = \frac{1}{W}(\dot{\vec{r}}_x, \dot{\vec{r}}_y, \ddot{\vec{r}}_{xx}) = \frac{1}{W}\begin{vmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & r \end{vmatrix} = \frac{r}{\sqrt{1 + p^2 + q^2}},$$

$$M = \frac{1}{W}(\dot{\vec{r}}_x, \dot{\vec{r}}_y, \ddot{\vec{r}}_{xy}) = \frac{1}{W}\begin{vmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & s \end{vmatrix} = \frac{s}{\sqrt{1 + p^2 + q^2}},$$

$$N = \frac{1}{W}(\dot{\vec{r}}_x, \dot{\vec{r}}_y, \ddot{\vec{r}}_{yy}) = \frac{1}{W}\begin{vmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & t \end{vmatrix} = \frac{t}{\sqrt{1 + p^2 + q^2}},$$

gdje su prema § 7. $p = \dfrac{\partial z}{\partial x}$, $\quad q = \dfrac{\partial z}{\partial y}$; $r = \dfrac{\partial^2 z}{\partial x^2}$, $s = \dfrac{\partial^2 z}{\partial x \partial y}$ i $t = \dfrac{\partial^2 z}{\partial y^2}$.

Druga diferencijalna forma, dakle, glasi:

$$\mathrm{II} = \frac{r\,\mathrm{d}x^2 + 2s\,\mathrm{d}x\,\mathrm{d}y + t\,\mathrm{d}y^2}{\sqrt{1 + p^2 + q^2}}.$$

(#) Naći drugu kvadratnu formu za rotacionu plohu

$$\vec{r} = \{\phi(u)\cos v,\ \phi(u)\sin v,\ \psi(u)\}.$$

Kako je

$$E = \phi'^2 + \psi'^2, \quad F = 0, \quad G = \phi^2,$$
$$W^2 = EG - F^2 = \phi^2(\phi'^2 + \psi'^2),$$

tamo imamo dalje:

$$L = \frac{1}{W}(\dot{\vec{r}}_u,\ \dot{\vec{r}}_v,\ \ddot{\vec{r}}_{uu}) = \frac{1}{W}\begin{vmatrix} \phi'\cos & \phi'\sin v & \psi' \\ -\phi\sin v & \phi\cos v & 0 \\ \phi''\cos v & \phi''\sin v & \psi'' \end{vmatrix};$$

$$L = \frac{\phi'\psi'' - \phi''\psi'}{\sqrt{\phi'^2 + \psi'^2}}.$$

$$M = \frac{1}{W}(\dot{\vec{r}}_u,\ \dot{\vec{r}}_v,\ \ddot{\vec{r}}_{uv}) = \frac{1}{W}\begin{vmatrix} \phi'\cos & \phi'\sin v & \psi' \\ -\phi\sin v & \phi\cos v & 0 \\ -\phi'\sin v & -\phi'\cos v & 0 \end{vmatrix};$$

$M = 0.$

$$N = \frac{1}{W}(\dot{\vec{r}}_u,\ \dot{\vec{r}}_v,\ \ddot{\vec{r}}_{vv}) = \frac{1}{W}\begin{vmatrix} \phi'\cos & \phi'\sin v & \psi' \\ -\phi\sin v & \phi\cos v & 0 \\ -\phi\cos v & -\phi\sin v & 0 \end{vmatrix};$$

$$N = \frac{\phi\psi'}{\sqrt{\phi'^2 + \psi'^2}}.$$

Druga diferencijalna forma glasi:

$$\mathrm{II} = \frac{(\phi'\psi'' - \phi''\psi')\,du^2 + \phi\psi'\,dv^2}{\sqrt{\phi'^2 + \psi'^2}}.$$